J. Math. Anal. Appl. 379 (2011) 649–655

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sandwich theorem for quasiconvex functions and its applications

Satoshi Suzuki [a,b,*], Daishi Kuroiwa [a,b]

[a] Interdisciplinary Graduate School of Science and Engineering, Shimane University, Japan
[b] Interdisciplinary Faculty of Science and Engineering, Shimane University, Japan

ARTICLE INFO

Article history:
Received 7 October 2010
Available online 31 January 2011
Submitted by H. Frankowska

Keywords:
Quasiconvex function
Sandwich theorem
Constraint qualification

ABSTRACT

In convex programming, sandwich theorem is very important because it is equivalent to Fenchel duality theorem. In this paper, we investigate a sandwich theorem for quasiconvex functions. Also, we consider some applications for quasiconvex programming.

1. Introduction

In convex programming, the following sandwich theorem was investigated, let f and g be proper lsc convex functions satisfying $f \geqslant -g$, and certain assumptions hold, then there exists an affine function K such that $f \geqslant K \geqslant -g$. Since sandwich theorem is equivalent to Fenchel duality theorem [1], sandwich theorem plays an important role in convex programming.

In this paper, we consider a sandwich theorem for quasiconvex functions. However, it is clear that even if f and g are quasiconvex functions satisfying $f \geqslant -g$, there does not always exist an affine function K such that $f \geqslant K \geqslant -g$. Hence, we consider a sufficient condition for the existence of a quasiaffine function K such that $f \geqslant K \geqslant -g$. Also, we investigate some applications of this sandwich theorem for quasiconvex programming.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we investigate sandwich theorem for quasiconvex functions. In Section 4, we show some applications of sandwich theorem in this paper. Finally, in Section 5, we compare our result with the sandwich theorem for convex functions.

2. Preliminaries

Let X be a locally convex Hausdorff topological vector space. In addition, let X^* be the continuous dual space of X, and let $\langle x^*, x\rangle$ denote the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $S \subset X^*$, we denote the w^*-closure, the convex hull, and the conical hull of S, by $\mathrm{cl}\, S$, $\mathrm{co}\, S$, and $\mathrm{cone}\, S$, respectively. The indicator function δ_A of A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \text{otherwise.} \end{cases}$$

Throughout the present paper, let f be a function from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Here, f is said to be proper if for all $x \in X$, $f(x) > -\infty$ and there exists $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by $\mathrm{dom}\, f$, that is, $\mathrm{dom}\, f =$

* Corresponding author at: Interdisciplinary Faculty of Science and Engineering, Shimane University, Japan.
E-mail address: suzuki@math.shimane-u.ac.jp (S. Suzuki).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.01.061

$\{x \in X \mid f(x) < \infty\}$. The epigraph of f, $\operatorname{epi} f$, is defined as $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leqslant r\}$, and f is said to be convex if $\operatorname{epi} f$ is convex. In addition, the Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) = \sup_{x \in \operatorname{dom} f}\{\langle u, x\rangle - f(x)\}$. Remember that f is said to be quasiconvex if for all $x_1, x_2 \in X$ and $\lambda \in (0, 1)$,

$$f\big((1-\lambda)x_1 + \lambda x_2\big) \leqslant \max\big\{f(x_1), f(x_2)\big\}.$$

Also, f is said to be quasiconcave if $-f$ is quasiconvex. Define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) = \big\{x \in X \bigm| f(x) \diamond \beta\big\}$$

for any $\beta \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\beta \in \mathbb{R}$, $L(f, \leqslant, \beta)$ is a convex set, or equivalently, for any $\beta \in \mathbb{R}$, $L(f, <, \beta)$ is a convex set. Any convex function is quasiconvex, but the opposite is not true.

It is well known that a proper lsc convex function consists of a supremum of some family of affine functions. In the case of quasiconvex functions, a similar result was also proved. First, we introduce the notion of a quasiaffine function which is a generalized notion of an affine function. A function f is said to be quasiaffine if quasiconvex and quasiconcave. It is worth noting that f is lsc quasiaffine if and only if there exist $k \in Q$ and $w \in X^*$ such that $f = k \circ w$, where $Q = \{h : \mathbb{R} \to \overline{\mathbb{R}} \mid h$ is lsc and nondecreasing$\}$. In [8], Penot and Volle proved that f is lsc quasiconvex if and only if there exists $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ such that $f = \sup_{i \in I} k_i \circ w_i$. This result indicates that an lsc quasiconvex function f consists of a supremum of some family of lsc quasiaffine functions. In [9], we define a notion of generator for quasiconvex functions, that is, $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ is said to be a generator of f if $f = \sup_{i \in I} k_i \circ w_i$. Because of the above Penot and Volle result, all lsc quasiconvex functions have at least one generator.

Moreover, we introduce a generalized notion of inverse function of $h \in Q$. The following function h^{-1} is said to be the hypo-epi-inverse of h:

$$h^{-1}(a) = \inf\big\{b \in \mathbb{R} \bigm| a < h(b)\big\} = \sup\big\{b \in \mathbb{R} \bigm| h(b) \leqslant a\big\}.$$

If h has an inverse function, then the inverse and the hypo-epi-inverse of h are the same, in detail, see [8]. In the present paper, we denote the hypo-epi-inverse of h by h^{-1}.

Recently, many researchers investigated constraint qualifications for Lagrange type duality theorems, see [5–7,9,10]. In [9], we investigated the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ). In this paper, we redefine the Q-CCCQ for infinitely constraints quasiconvex programming.

Definition 1. (See [9].) Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(h_{(i,j)}, u_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i for each $i \in I$, and $T = \{t = (i, j) \mid i \in I, j \in J_i\}$. Assume that $A = \{x \in X \mid \forall i \in I,\ g_i(x) \leqslant 0\}$ is nonempty set. Then, the quasiconvex system $\{g_i(x) \leqslant 0 \mid i \in I\}$ satisfies the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) w.r.t. $\{(h_t, u_t) \mid i \in T\}$ if

$$\operatorname{cone} \operatorname{co} \bigcup_{t \in T} \big\{(u_t, \delta) \in X^* \times \mathbb{R} \bigm| h_t^{-1}(0) \leqslant \delta\big\} + \{0\} \times [0, \infty)$$

is w^*-closed.

Also, $\{g_i(x) \leqslant 0 \mid i \in I\}$ satisfies the Q-CCCQ if and only if the alternative form of the Q-CCCQ

$$\operatorname{epi} \delta_A^* \subset \operatorname{cone} \operatorname{co} \bigcup_{t \in T} \big\{(u_t, \delta) \in X^* \times \mathbb{R} \bigm| h_t^{-1}(0) \leqslant \delta\big\} + \{0\} \times [0, \infty)$$

holds.

A subset B of X is said to be evenly convex if B is equal to the intersection of some family of open halfspaces, in detail, see [3,4]. A function f is said to be evenly quasiconvex if for each $\beta \in \mathbb{R}$, $L(f, \leqslant, \beta)$ is evenly quasiconvex, and f is said to be evenly quasiaffine if evenly quasiconvex and quasiconcave. In [8], Penot and Volle investigated that f is evenly quasiaffine if and only if there exist $k \in G$ and $w \in X^*$ such that $f = k \circ w$, where $G = \{h : \mathbb{R} \to \overline{\mathbb{R}} \mid h$ is nondecreasing$\}$. Also, $G_{\mathbb{R}}$ denotes the set of all real-valued nondecreasing functions, that is, $G_{\mathbb{R}} = \{h : \mathbb{R} \to \mathbb{R} \mid h$ is nondecreasing$\}$. The following proposition is important.

Proposition 1. *(See [8].) The following* (i) *and* (ii) *are equivalent*:

(i) *f is evenly quasiaffine,*
(ii) *for each $\beta \in \mathbb{R}$, $L(f, \leqslant, \beta)$ is an open or closed halfspace, or X, or $\emptyset$.*

3. Sandwich theorem

In this section, we show a sandwich theorem for quasiconvex functions. In convex case, if f and g are convex, $f \geqslant -g$, and certain assumptions hold, then there exists an affine function K such that $f \geqslant K \geqslant -g$. In this paper, we consider a sufficient condition for the existence of a real-valued evenly quasiaffine function K such that $f \geqslant K \geqslant -g$. Now we show some lemmas.

Lemma 1. *Let f be a quasiconvex function with a generator $\{(k_i, w_i) \mid i \in I\} \subset G_{\mathbb{R}} \times X^*$. If $0 \in \operatorname{co}\{w_i \mid i \in I\}$, then f is bounded from below.*

Proof. If $0 \in \operatorname{co}\{w_i \mid i \in I\}$, then there exist $m \in \mathbb{N}$, $i_1, \ldots, i_m \in I$ and $\beta_1, \ldots, \beta_m \geqslant 0$ such that $0 = \sum_{n=1}^m \beta_n w_{i_n}$ and $1 = \sum_{n=1}^m \beta_n$. Then, for all $x \in X$, there exists $n_0 \in \{1, \ldots, m\}$ such that $\langle w_{i_{n_0}}, x\rangle \geqslant 0$. Hence, $f(x) \geqslant k_{i_{n_0}}(\langle w_{i_{n_0}}, x\rangle) \geqslant k_{i_{n_0}}(0) \geqslant \min_{n=1,\ldots,m} k_{i_n}(0)$. This completes the proof. $\square$

Lemma 2. *Let f be a proper quasiconvex function with a generator $\{(k_i, w_i) \mid i \in I\} \subset G_{\mathbb{R}} \times X^*$. If $v \in \operatorname{cone}\operatorname{co}\{w_i \mid i \in I\}$, then the function $K_{(f,v)}$ on X defined by*

$$K_{(f,v)}(x) = \inf\bigl\{ f(z) \bigm| \langle v, z\rangle \geqslant \langle v, x\rangle\bigr\}$$

is proper.

Proof. If $v \in \operatorname{cone}\operatorname{co}\{w_i \mid i \in I\}$, then there exist $m \in \mathbb{N}$, $i_1, \ldots, i_m \in I$, $\lambda_1, \ldots, \lambda_m \geqslant 0$ such that $v = \sum_{n=1}^m \lambda_n w_{i_n}$. For all $z \in X$ with $\langle v, z\rangle \geqslant \langle v, x\rangle$, $\langle \sum_{n=1}^m \lambda_n w_{i_n}, z\rangle \geqslant \langle \sum_{n=1}^m \lambda_n w_{i_n}, x\rangle$, that is, there exists $n_0 \in \{1, \ldots, m\}$ such that $\langle w_{i_{n_0}}, z\rangle \geqslant \langle w_{i_{n_0}}, x\rangle$. Then,

$$\max_{n=1,\ldots,m} k_{i_n} \circ w_{i_n}(z) \geqslant k_{i_{n_0}} \circ w_{i_{n_0}}(z) \geqslant k_{i_{n_0}} \circ w_{i_{n_0}}(x) \geqslant \min_{n\in\{1,\ldots,m\}} k_{i_n} \circ w_{i_n}(x),$$

that is, $K_{(f,v)}(x) \geqslant \min_{n\in\{1,\ldots,m\}} k_{i_n} \circ w_{i_n}(x) > -\infty$. Since $f \geqslant K_{(f,v)}$ and $\operatorname{dom} f$ is nonempty, $K_{(f,v)}$ is proper. $\square$

Lemma 3. *$K_{(f,v)}(x) = \inf\{f(z) \mid \langle v, z\rangle \geqslant \langle v, x\rangle\}$ is an evenly quasiaffine function.*

Proof. We show that $L(K_{(f,v)}, \leqslant, \beta)$ is an open or closed halfspace, or X, or $\emptyset$ for each $\beta \in \mathbb{R}$. If $L(K_{(f,v)}, \leqslant, \beta)$ is a nonempty and proper subset of X, then it is clear that $L(K_{(f,v)}, \leqslant, \beta) \subset \{x \mid \langle v, x\rangle \leqslant \sup_{y\in L(K_{(f,v)},\leqslant,\beta)}\langle v, y\rangle\}$, and we can check that $\delta^*_{L(K_{(f,v)},\leqslant,\beta)}(v) \in \mathbb{R}$. If there exists $y_0 \in L(K_{(f,v)}, \leqslant, \beta)$ such that $\langle v, y_0\rangle = \sup_{y\in L(K_{(f,v)},\leqslant,\beta)}\langle v, y\rangle$, we can check that $L(K_{(f,v)}, \leqslant, \beta) = \{x \in X \mid \langle v, x\rangle \leqslant \langle v, y_0\rangle\}$ by the definition of $K_{(f,v)}$. Also, if for all $x \in L(K_{(f,v)}, \leqslant, \beta)$, $\langle v, x\rangle < \sup_{y\in L(K_{(f,v)},\leqslant,\beta)}\langle v, y\rangle$, we can check that $L(K_{(f,v)}, \leqslant, \beta) = \{x \mid \langle v, x\rangle < \sup_{y\in L(K_{(f,v)},\leqslant,\beta)}\langle v, y\rangle\}$. Since Proposition 1, $K_{(f,v)}$ is evenly quasiaffine. $\square$

Lemma 4. *Let f be a quasiconvex function with a generator $\{(k_i, w_i) \mid i \in I\} \subset G \times X^*$, and g be a quasiconvex function with a generator $\{(h_j, u_j) \mid j \in J\} \subset G \times X^*$. Assume that $f \geqslant -g$ and $B = \operatorname{co}\{x - y \mid f(x) + g(y) < 0\}$ is nonempty. Then $B^* \subset \operatorname{cl}\operatorname{cone}\operatorname{co}\{w_i \mid i \in I\} \cap \operatorname{cl}\operatorname{cone}\operatorname{co}\{-u_j \mid j \in J\}$, where B^* is the negative polar cone of B.*

Proof. Since $B \neq \emptyset$, there exist x_0 and $y_0 \in X$ such that $f(x_0) + g(y_0) < 0$. Assume that $v \notin \operatorname{cl}\operatorname{cone}\operatorname{co}\{w_i \mid i \in I\}$. By using separation theorem, there exists $x \in X$ such that for all $i \in I$, $\langle v, x\rangle > 0 \geqslant \langle w_i, x\rangle$. Hence, for all $n \in \mathbb{N}$ and $i \in I$, $k_i \circ w_i(x_0 + nx) \leqslant k_i \circ w_i(x_0) \leqslant f(x_0)$, that is, $x_0 + nx - y_0 \in B$. However, $\langle v, x_0 + nx - y_0\rangle$ diverges to infinity, this shows that $v \notin B^*$. We can prove similarly that $B^* \subset \operatorname{cl}\operatorname{cone}\operatorname{co}\{-u_j \mid j \in J\}$. $\square$

Consider the following set of functions:

$$\varXi(X) = \left\{ \sup_{i\in I} k_i \circ w_i \;\middle|\; \begin{array}{l} \{(k_i, w_i) \mid i \in I\} \subset G_{\mathbb{R}} \times X^* \\ \operatorname{co}\{w_i \mid i \in I\}\text{: } w^*\text{-compact} \end{array} \right\}.$$

For example, if I is finite, or I is a compact topological space, $\{w_i \mid i \in I\}$ is convex, and w_i is w^*-continuous on I, then $f = \sup_{i\in I} k_i \circ w_i \in \varXi(X)$.

Now we prove the following sandwich theorem for quasiconvex functions.

Theorem 1. *Let $f, g \in \varXi(X)$ be proper, $f = \sup_{i\in I} k_i \circ w_i$, $g = \sup_{j\in J} h_j \circ u_j$, at least one of f and g be usc, and $f \geqslant -g$. Assume that $0 \notin B = \operatorname{co}\{x - y \mid f(x) + g(y) < 0\}$. Then, there exists a real-valued evenly quasiaffine function K such that $f \geqslant K \geqslant -g$.*

Proof. At first, we can check that B is open convex since at least one of f and g is usc. If B is empty set, then it is clear that there exists $\beta \in \mathbb{R}$ such that $f \geqslant \beta \geqslant -g$. If B is nonempty, then by using separation theorem, there exists $v \in X^* \setminus \{0\}$ such that for all $y \in B$,

$$\langle v, 0\rangle = 0 > \langle v, y\rangle,$$

that is, $v \in B^*$. Then, $K_{(f,v)}$ is proper. Actually, if $0 \in \operatorname{co}\{w_i \mid i \in I\}$, by using Lemma 1, f is bounded from below, that is $K_{(f,v)}$ is proper. If $0 \notin \operatorname{co}\{w_i \mid i \in I\}$, $\operatorname{cone}\operatorname{co}\{w_i \mid i \in I\}$ is w^*-closed since $\operatorname{co}\{w_i \mid i \in I\}$ is w^*-compact. By using Lemma 2 and 4, $K_{(f,v)}$ is proper. We can prove similarly that $K_{(g,-v)}$ is proper evenly quasiaffine. Next, we show that $K_{(f,v)} \geqslant -K_{(g,-v)}$. If there exists $x \in X$ such that $K_{(f,v)}(x) < -K_{(g,-v)}(x)$, then there exist $x_0, z_0 \in X$ and $\lambda \in \mathbb{R}$ such that $f(x_0) < \lambda < -g(z_0)$ and $\langle v, x_0\rangle \geqslant \langle v, x\rangle \geqslant \langle v, z_0\rangle$. Hence, $x_0 - z_0 \in B$ and $\langle v, x_0 - z_0\rangle \geqslant 0$, this is a contradiction. Therefore, $f \geqslant K_{(f,v)} \geqslant -K_{(g,-v)} \geqslant -g$. Since $K_{(f,v)}$ is proper, there exists $x_0 \in X$ such that $K_{(f,v)}(x_0) \in \mathbb{R}$. Put K as follows:

$$K(x) := \begin{cases} K_{(f,v)}(x), & \langle v, x\rangle \leqslant \langle v, x_0\rangle, \\ \max\{K_{(f,v)}(x_0), -K_{(g,-v)}(x)\}, & \text{otherwise}, \end{cases}$$

then we can check that K is real-valued evenly quasiaffine, and $K_{(f,v)} \geqslant K \geqslant -K_{(g,-v)}$. This completes the proof. □

4. Applications

In this section we show some applications of Theorem 1, and we investigate the relation between sandwich theorem and the Q-CCCQ in [9].

Theorem 2. *Let A be a nonempty closed convex subset of X, $f \in \Xi(X)$ be usc, and $\alpha = \inf_{x\in A} f(x) \in \mathbb{R}$. Then, there exists a real-valued evenly quasiaffine function K such that*

(i) $f \geqslant K \geqslant \alpha - \delta_A$,
(ii) $\inf_{x\in A} f(x) = \inf_{x\in A} K(x)$, *and*
(iii) $\inf_{x\in X}\{f(x) - K(x)\} = 0$.

Proof. At first, we apply Theorem 1 with $g = \delta_A - \alpha$. Let $B = \operatorname{co}\{x - y \mid f(x) + \delta_A(y) - \alpha < 0\}$. Put an open line segment $L = (\inf_{x\in X} f(x), \alpha)$, then we can check that $B = \operatorname{co}\bigcup_{\lambda\in L}\{L(f, <, \lambda) - A\}$. When $L = \emptyset$, it is clear that $0 \notin B$ since $B = \emptyset$. Assume that $L \neq \emptyset$ and $0 \in B$. Then, there exist $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in L$, $x_1, \dots, x_m, y_1, \dots, y_m \in X$ and $\beta_1, \dots, \beta_m \geqslant 0$ such that $0 = \sum_{n=1}^m \beta_n(x_n - y_n)$, $x_n \in L(f, <, \lambda_n)$ and $y_n \in A$ for each $n \in \{1, \dots, m\}$, and $1 = \sum_{n=1}^m \beta_n$. Since A is convex and $\lambda_n \in L$, for each n,

$$\begin{aligned} \alpha &\leqslant f\Big(\sum \beta_n x_n\Big) + \delta_A\Big(\sum \beta_n y_n\Big) \\ &\leqslant \max_{n=1,\dots,m} f(x_n) + 0 \\ &\leqslant \max_{n=1,\dots,m} \lambda_n \\ &< \alpha. \end{aligned}$$

This is a contradiction. Since $f \geqslant \alpha - \delta_A$ and $0 \notin B$, by using Theorem 1, there exists a real-valued evenly quasiaffine function K such that $f \geqslant K \geqslant \alpha - \delta_A$. Hence, $\inf_{x\in A} f(x) \geqslant \inf_{x\in X}\{f(x) - K(x)\} + \inf_{x\in A} K(x) \geqslant \inf_{x\in A} K(x) \geqslant \inf_{x\in A}\{\alpha - \delta_A(x)\} = \alpha = \inf_{x\in A} f(x)$. This completes the proof. □

Next, we consider an optimization problem with quasiconvex inequality constraints. For the sake of simplicity, we consider the problem with singular constraint function.

Theorem 3. *Let g be an lsc quasiconvex function from X to $\overline{\mathbb{R}}$, and $\{(h_t, u_t) \mid t \in T\} \subset Q \times X^*$ be a generator of g. Assume that $A = L(g, \leqslant, 0)$ is nonempty. Then the following conditions* (i) *and* (ii) *are equivalent.*

(i) $\{g(x) \leqslant 0\}$ *satisfies the Q-CCCQ w.r.t.* $\{(h_t, u_t) \mid t \in T\}$,
(ii) *for all usc function $f \in \Xi(X)$ with $\alpha = \inf_{x\in A} f(x) \in \mathbb{R}$, there exist $k_0 \in G_{\mathbb{R}}$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$\begin{cases} \inf_{x\in A} f(x) = \inf_{x\in X}\left\{ f(x) - k_0\left(\left\langle -\sum_{t\in T}\lambda_t u_t, x\right\rangle\right)\right\}, \\ \inf_{x\in A} k_0\left(\left\langle -\sum_{t\in T}\lambda_t u_t, x\right\rangle\right) = 0, \\ \delta_A^*\left(\sum_{t\in T}\lambda_t u_t\right) = \sum_{t\in T}\lambda_t h_t^{-1}(0). \end{cases}$$

Proof. We prove that (i) implies (ii). Let $f \in \Xi(X)$ and $\alpha = \inf_{x\in A} f(x) \in \mathbb{R}$. Then there exist $k_0 \in G_{\mathbb{R}}$ and $w_0 \in X^*$ such that

$$\alpha = \inf_{x\in X}\{f(x) - k_0 \circ w_0(x)\} + \inf_{y\in A} k_0 \circ w_0(y) \quad \text{and} \quad \delta_A^*(-w_0) \in \mathbb{R}. \tag{1}$$

Actually, by using Theorem 2, there exists a real-valued evenly quasiaffine function K such that $f \geqslant K \geqslant \alpha - \delta_A$, $\inf_{x\in X}\{f(x) - K(x)\} = 0$, and $\alpha = \inf_{y\in A} K(y)$. Since K is evenly quasiaffine, there exist $k_0 \in G_{\mathbb{R}}$ and $w_0 \in X^*$ such that $K = k_0 \circ w_0$. If $\delta_A^*(-w_0) \in \mathbb{R}$, then (1) holds. Assume that $\delta_A^*(-w_0) \notin \mathbb{R}$, we show $f \geqslant \alpha \geqslant \alpha - \delta_A$. It is clear that $\delta_A^*(-w_0) = \infty$ and $\alpha \geqslant \alpha - \delta_A$. Since $\inf_{x\in A}\langle w_0, x\rangle = -\delta_A^*(-w_0) = -\infty$ and $k_0 \in G$, we can check that $\inf_{t\in\mathbb{R}} k_0(t) = \inf_{x\in A} k_0 \circ w_0(x) = \alpha$. Hence, for all $x \in X$,

$$f(x) \geqslant k_0 \circ w_0(x) \geqslant \inf_{t\in\mathbb{R}} k_0(t) = \alpha,$$

that is, $f \geqslant \alpha$. Now we replace $k_0 \equiv \alpha$ and $w_0 = 0$, then (1) is satisfied.

Since $(-w_0, \delta_A^*(-w_0)) \in \operatorname{epi}\delta_A^*$ and the Q-CCCQ is satisfied, there exist $\lambda \in \mathbb{R}_+^{(T)}$, $\delta \in \mathbb{R}^T$ and $r \geqslant 0$ such that $-w_0 = \sum_{t\in T}\lambda_t u_t$, $\delta_t \geqslant h_t^{-1}(0)$, and $\delta_A^*(-w_0) = \sum_{t\in T}\lambda_t\delta_t + r$. By the similar way in [9], we can check that $\delta_t = h_t^{-1}(0)$ and $r = 0$, that is, $\delta_A^*(-w_0) = \sum_{t\in T}\lambda_t h_t^{-1}(0)$. Also, we replace k_0 as $k_0 - \inf_{x\in A} k_0(\langle -\sum_{t\in T}\lambda_t u_t, x\rangle)$, then condition (ii) holds.

Next, we prove that (ii) implies (i). We may show that for all $v \in \operatorname{dom}\delta_A^* \setminus \{0\}$,

$$\left(v, \delta_A^*(v)\right) \in \operatorname{cone}\operatorname{co}\bigcup_{t\in T}\left\{(u_t, \delta) \in X^* \times \mathbb{R} \;\middle|\; h_t^{-1}(0) \leqslant \delta\right\}.$$

Let $v \in \operatorname{dom}\delta_A^* \setminus \{0\}$, then $\inf_{x\in A}\langle -v, x\rangle = \delta_A^*(v) \in \mathbb{R}$. By using (ii), there exist $k_0 \in G_{\mathbb{R}}$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that $\inf_{x\in A}\langle -v, x\rangle = \inf_{x\in X}\{\langle -v, x\rangle - k_0(\langle -\sum_{t\in T}\lambda_t u_t, x\rangle)\}$, $\inf_{y\in A} k_0(\langle -\sum_{t\in T}\lambda_t u_t, y\rangle) = 0$ and $\delta_A^*(\sum_{t\in T}\lambda_t u_t) = \sum_{t\in T}\lambda_t h_t^{-1}(0)$. Then, we can prove that $v \in \mathbb{R}_+\{\sum_{t\in T}\lambda_t u_t\}$. At first, we assume that $v \notin \mathbb{R}\{\sum_{t\in T}\lambda_t u_t\}$, then there exists $x_0 \in X$ such that for all $a \in \mathbb{R}$, $\langle v, x_0\rangle > \langle a\sum_{t\in T}\lambda_t u_t, x_0\rangle$ by separation theorem. This implies that $\langle v, x_0\rangle > 0 = \langle \sum_{t\in T}\lambda_t u_t, x_0\rangle$. However, $\inf_{x\in X}\{\langle -v, x\rangle - k_0(\langle -\sum_{t\in T}\lambda_t u_t, x\rangle)\} \leqslant \inf_{a\in\mathbb{R}}\{\langle -v, ax_0\rangle - k_0(\langle -\sum_{t\in T}\lambda_t u_t, ax_0\rangle)\} = -\infty$, this is a contradiction. Hence, there exists $\gamma \neq 0$ such that $v = \gamma\sum_{t\in T}\lambda_t u_t$ and we can choose $y_0 \in X$ such that $\langle v, y_0\rangle > 0$ since $v \neq 0$. If $\gamma < 0$,

$$\begin{aligned} \inf_{x\in X}\left\{\langle -v, x\rangle - k_0\left(\left\langle -\sum_{t\in T}\lambda_t u_t, x\right\rangle\right)\right\} &= \inf_{x\in X}\left\{\langle -v, x\rangle - k_0\left(\left\langle -\frac{v}{\gamma}, x\right\rangle\right)\right\} \\ &\leqslant \inf_{a\geqslant 0}\left\{\langle -v, ay_0\rangle - k_0\left(\left\langle -\frac{v}{\gamma}, ay_0\right\rangle\right)\right\} \\ &\leqslant \inf_{a\geqslant 0}\left\{\langle -v, ay_0\rangle\right\} - k_0(0) \\ &= -\infty. \end{aligned}$$

Therefore, $\gamma > 0$. Now we put $\bar\lambda = \gamma\lambda$, then it is clear that $\bar\lambda \in \mathbb{R}_+^{(T)}$, $v = \sum_{t\in T}\bar\lambda_t u_t$, and $\delta_A^*(v) = \sum_{t\in T}\bar\lambda_t h_t^{-1}(0)$. This completes the proof. □

5. Discussion

In this section, we compare Theorem 1 with the sandwich theorem for convex functions. It is known that '$0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$' and '$\operatorname{epi} f^* + \operatorname{epi} g^*$ is w^*-closed' are sufficient conditions for sandwich theorem for convex functions, in detail, see [1,2]. In Theorem 1, we propose the following sufficient condition for sandwich theorem:

(1) $0 \notin B = \operatorname{co}\{x - y \mid f(x) + g(y) < 0\}$.

Also, in Theorem 2, we show that a usc function $f \in \Xi(X)$ and $\delta_A - \alpha$ satisfy condition (1) where A is a nonempty closed convex subset of X and $\alpha = \inf_{x\in A} f(x) \in \mathbb{R}$.

The following theorem indicates that two convex functions satisfy condition (1).

Theorem 4. *Let f and g be convex functions from X to $\overline{\mathbb{R}}$ and $f \geqslant -g$. Then, f and g satisfy condition* (1).

Proof. Assume that $0 \in B$. Then, there exist $m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \geqslant 0$, $x_1, \ldots, x_m \in X$, $y_1, \ldots, y_m \in X$ such that

$$\begin{cases} \sum_{i=1}^{m} \beta_i = 1, \\ 0 = \sum_{i=1}^{m} \beta_i (x_i - y_i), \\ f(x_i) + g(y_i) < 0, \quad \forall i \in \{1, \ldots, m\}. \end{cases}$$

Put $x_0 = \sum_{i=1}^{m} \beta_i x_i$, then

$$\begin{aligned} 0 &\leqslant f(x_0) + g(x_0) \\ &= f\left(\sum_{i=1}^{m} \beta_i x_i\right) + g\left(\sum_{i=1}^{m} \beta_i y_i\right) \\ &\leqslant \sum_{i=1}^{m} \beta_i f(x_i) + \sum_{i=1}^{m} \beta_i g(y_i) \\ &= \sum_{i=1}^{m} \beta_i \big(f(x_i) + g(y_i)\big) \\ &< 0. \end{aligned}$$

This is a contradiction. □

Hence, we can prove the following corollary.

Corollary 1. *Let $f, g \in \Xi(X)$ be proper convex, at least one of f and g is usc, and $f \geqslant -g$. Then, there exists a real-valued evenly quasiaffine function K such that $f \geqslant K \geqslant -g$.*

Although Corollary 1 does not guarantee the existence of an affine function, Corollary 1 indicates that if f and $g \in \Xi(X)$ are proper usc convex with $f \geqslant -g$, then there exists a quasiaffine function K such that $f \geqslant K \geqslant -g$ without any other sufficient condition of the sandwich theorem for convex functions. The following example shows this situation.

Example 1. Let f and g be convex functions from $\mathbb{R}$ to $\overline{\mathbb{R}}$ as follows.

$$f(x) := \begin{cases} -\sqrt{|x^2 + 2x|}, & x \in [-2, 0), \\ \infty, & \text{otherwise}, \end{cases}$$

$$g(x) := \begin{cases} -\sqrt{|x^2 - 2x|}, & x \in [0, 2], \\ \infty, & \text{otherwise}. \end{cases}$$

Then, $f \geqslant -g$, $f, g \in \Xi(X)$ and f is usc. Also, we can check that there does not exist an affine function K such that $f \geqslant K \geqslant -g$. However, by Corollary 1, there exists a real-valued evenly quasiaffine function K such that $f \geqslant K \geqslant -g$. Actually, the following K satisfies $f \geqslant K \geqslant -g$.

$$K(x) := \begin{cases} -1, & x \leqslant 1, \\ -\sqrt{|x^2 + 2x|}, & x \in [-1, 0], \\ \sqrt{|x^2 - 2x|}, & x \in [0, 1], \\ 1, & x \geqslant 1. \end{cases}$$

Acknowledgments

The authors are grateful to anonymous referee for many comments and suggestions improved the quality of the paper.

References

[1] R.M. Borwein, Q.J. Zhu, Variational methods in convex analysis, J. Global Optim. 35 (2006) 197–213.
[2] R.S. Burachik, V. Jeyakumar, A new geometric condition for Fenchel's duality in infinite dimensional spaces, Math. Program. 104 (2005) 229–233.
[3] A. Daniilidis, J.-E. Martinez-Legaz, Characterizations of evenly convex sets and evenly quasiconvex functions, J. Math. Anal. Appl. 273 (2002) 58–66.
[4] W. Fenchel, A remark on convex sets and polarity, Comm. Sem. Math. Univ. Lund Tome Suppl. (1952) 82–89.
[5] M.A. Goberna, V. Jeyakumar, M.A. López, Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities, Nonlinear Anal. 68 (2008) 1184–1194.
[6] V. Jeyakumar, N. Dinh, G.M. Lee, A new closed cone constraint qualification for convex optimization, Research Report AMR 04/8, Department of Applied Mathematics, University of New South Wales, 2004.
[7] C. Li, K.F. Ng, T.K. Pong, Constraint qualifications for convex inequality systems with applications in constrained optimization, SIAM J. Optim. 19 (2008) 163–187.
[8] J.P. Penot, M. Volle, On quasi-convex duality, Math. Oper. Res. 15 (1990) 597–625.
[9] S. Suzuki, D. Kuroiwa, On set containment characterization and constraint qualification for quasiconvex programming, J. Optim. Theory Appl., doi:10.1007/s10957-011-9804-8, in press.
[10] S. Suzuki, D. Kuroiwa, Optimality conditions and the basic constraint qualification for quasiconvex programming, Nonlinear Anal. 74 (2011) 1279–1285.